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On Retrogression in Mean Ergodic Theory

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1. INTRODUCTION

"Those who cannot remember the past are condemned to repeat it" (George Santayana [7]). In 1949 we published in an unobscure journal (*Trans. Amer. Math. Soc.*) a paper [2] on mean ergodic theory and weak almost periodicity (in which we introduced the latter concept). Hardly a year has passed since then without seeing the publication of *special cases* of our results, and even *rediscoveries* of special cases. (This may be partly because *Math. Rev.* did not review the ergodic theory part of our 1949 paper and partly because, contrary to popular belief, mathematicians tend not to read each others' papers.) Our one comfort was that good friends assured us our paper was known in Paris at least. Now, alas, it appears that "the centre cannot hold" [8]: Gustave Choquet has just communicated to the Paris Academy of Science a paper on mean ergodic theory by Richard Emilion [4], who seems unaware of American work in the subject.

In this paper we shall first recall our abstract mean ergodic theorem of 1949 and then show how it specializes to yield some later results of others, ending up with stronger version of the basic theorems of Emilion [4].

2. THE ABSTRACT MEAN ERGODIC THEOREM

We first specialize our mean ergodic theorem from an arbitrary locally convex linear topological space to a normed linear space **E**, not necessarily complete. Let $\mathbf{\Gamma}$ be a set of bounded linear transformations in **E** and $\mathbf{S} = \boldsymbol{S}(\mathbf{\Gamma})$ the semigroup generated by I and $\mathbf{\Gamma}$.

DEFINITION. A system of almost invariant integrals for Γ is a net (T_α) of bounded linear transformations in E such that

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(I) *For every x and α $T_\alpha(x) \in K(x)$*, where $K(x)$ is the closed convex hull of Sx.

(II) $M \equiv \sup_\alpha \|T_\alpha\| < \infty$.

(III) For every x in E and T in Γ,

$\lim_\alpha (TT_\alpha - T_\alpha)x = 0 = \lim_\alpha (T_\alpha T - T_\alpha)x$.

Our basic mean ergodic theorem of 1949 then specializes to

THEOREM 1. *Let (T_α) be ANY system of almost invariant integrals for Γ and let $x \in E$. Then the following conditions on an element y in E are equivalent*:

(1) *$y \in K(x)$ and $Ty = y$ for all T in Γ*;

(2) $y = \lim_\alpha T_\alpha x$;

(3) *$y = \lim_\alpha T_\alpha x$ weakly*;

(4) *y is a weak cluster point of $(T_\alpha x)$.*

Call **x** ergodic with (unique) limit fixed point **y** if there exists a $y = T_\infty x$ satisfying any of the conditions (1)–(4). When **E** is reflexive clearly every **x** is ergodic, since the bounded set $(T_\alpha x)$ is conditionally weakly compact.

Remark. One need only assume that **Γ** possesses at least one system of almost invariant integrals. *All systems are equivalent since* $\lim_\alpha T_\alpha x = \lim_\beta T_\beta x$ *in the sense that if either limit exists so does the other and the two limits are equal.* Call **Γ** ergodic if it possesses at least one system of almost invariant integrals.

COROLLARY. *If Γ is ergodic, the ergodic elements of E constitute a closed invariant subspace Ω. The transformation $T_\infty x = y = \lim_\alpha T_\alpha x$ is a bounded linear transformation of Ω into itself and $\|T_\infty\|_\Omega \leqslant M$. On Ω, $T_\infty = T_\infty^2 = T_\infty U = UT_\infty$ for every U in S. $\Omega = [x: Tx = x, T \in \Gamma] + \Sigma_{T \in S}(I - T)E$.*

The last equality was omitted from our 1949 paper but is trivial: If x is ergodic, $x = T_\infty x + (I - T_\infty)x = T_\infty x + \lim_\alpha (I - T_\alpha)x$. It follows from (I) that $T_\alpha x = \lim_n V_n x$, where V_n has the form

$$V_n = \sum_{j=1}^{N(n)} a_j T_j, \qquad \text{where all} \quad a_j \geqslant 0, \ \sum_1^{N(n)} a_j = 1,$$

and all $T_j \in S$. Hence

$$(I - T_\alpha)x = \lim_n \sum_1^{N(n)} (I - T_j)\, a_j x.$$

Conversely, if

$$x = y + \lim_n \sum_1^{N(n)} (I - T_j) x_j \qquad (T_j \in S),$$

where y is a fixed point under Γ, clearly $\lim_\alpha T_\alpha x = y$.

The last statement of the Corollary implies Emilion's Corollary 4.3.

3. Cesaro and Other Means

Let Γ consist of a single bounded operator T and set $T_n = n^{-1} \sum_0^{n-1} T^k$. Then condition (I) is trivially fulfilled and $TT_n - T_n = T_n T - T_n = n^{-1}(T^n - I)$. Thus both conditions (II) and (III) are satisfied as $n \to \infty$ if $\sup_n \|T^n\| < \infty$. But this last hypothesis is needlessly strong: As pointed out in examples by Dunford [1] and Hille [5] and by us in [2], all one needs are

(II′) $M \equiv \sup_n \|T_n\| < \infty$;

(III′) $\lim_n n^{-1} T^n x = 0$ $(x \in E)$.

Emilion [4] attributes condition (II′) to A. Brunel (without reference).

Various authors have replaced $(C, 1)$ means by other summability methods. Given a real infinite matrix $A = (a_{mn})$ $(m, n = 0, 1, \ldots)$, one sets $U_m = \sum_{n=0}^\infty a_{mn} T^n$. When A is row finite the U_m exist trivially. In the general case one assumes: E is complete, some type of boundedness condition on the (T^n), and the following positive type Toeplitz conditions:

$$\sum_{n=0}^\infty a_{mn} = 1 \qquad (m = 0, 1, \ldots); \tag{1}$$

$$a_{mn} \geqslant 0 \qquad (m, n = 0, 1, \ldots). \tag{2}$$

These conditions plus the classical hypothesis $N \equiv \sup_n \|T^n\| < \infty$ are sufficient to imply: the existence of the (U_m) in the uniform operator topology, condition (I), and $\sup_m \|U_m\| \leqslant N$ (II). The validity of condition (III) hinges on the identity $TU_m - U_m = U_m T - U_m = \sum_{n=0}^\infty (a_{mn} - a_{m,n+1}) T^{n+1} - a_{m0} I$. Given that $N < \infty$, an obvious *sufficient* condition for the validity of (III) is that

$$\lim_m \sum_{n=0}^\infty |a_{mn} - a_{m,n+1}| = 0, \tag{3}$$

one trivially satisfied by the $(C, 1)$ matrix. Clearly (3) implies $\lim_m a_{mn} = 0$ $(n = 0, 1, \ldots)$, the remaining Toeplitz condition for regularity of the summability method **A**.

Dropping the positivity condition is a sterile generalization, since the case of a nonpositive Toeplitz matrix $A = (a_{mn})$ reduces to the positive case as follows: One can still assume without loss of generality that (1) holds, but (2) is replaced by

$$\|A\| \equiv \sup_m \sum_{n=0}^{\infty} |a_{mn}| < \infty. \tag{2'}$$

Set $d_m = \sum_{n=0}^{\infty} |a_{mn}|$. Then $1 \leqslant d_m \leqslant \|A\|$, and one writes $a_{mn} = \frac{1}{2}(d_m + 1)\, b_{mn} - \frac{1}{2}(d_m - 1)\, c_{mn}$, where $b_{mn} = (|a_{mn}| + a_{mn})/(d_m + 1)$ for all m and $c_{mn} = (|a_{mn}| - a_{mn})/(d_m - 1)$ when $d_m > 1$. When $d_m = 1$, $a_{mn} \geqslant 0$ $(n = 0, 1,...)$ and one arbitrarily sets $c_{mn} = h_{mn}$, where (h_{mn}) is the $(C, 1)$ matrix. Clearly the matrices (b_{mn}) and (c_{mn}) satisfy conditions (1) and (2), plus (3) if **A** does. Moreover $\lim_m \sum_{n=0}^{\infty} b_{mn} T^n x = y = \lim_m \sum_{n=0}^{\infty} c_{mn} T^n x$ implies $\lim_m \sum_{n=0}^{\infty} a_{mn} T^n x = y$.

Generalizations to other summability methods are usually specious: Given any method **A** of positive type one need only verify whether the associated means (U_m) form a system of almost invariant integrals. Since all such systems are equivalent, why introduce new ones?

4. Abel Means

Assume henceforth that **E** is a Banach space and that (II$'$) $M \equiv \sup_n \|T_n\| < \infty$. Consider the Abel means

$$A(\lambda) = \lambda \sum_{0}^{\infty} T^k/(\lambda + 1)^{k+1} \qquad (\lambda > 0).$$

Theorem 2. (Emilion's Theorem 1.3). *The Abel series $A(\lambda)$ converge in the uniform operator topology and $\|A(\lambda)\| \leqslant M$ $(\lambda > 0)$.*

Since $B(\mathbf{E})$, the algebra of bounded linear transformations in **E**, is a Banach algebra under the operator norm, Theorem 2 is a special case of the following folk result:

Vector Abelian Theorem. *Let $x_0, x_1, x_2,...,$ be a sequence in a Banach space and let $\sigma_n = n^{-1}(x_0 + \cdots + x_{n-1})$ be the associated sequence of arithmetic means. Assume $M \equiv \sup_n \|\sigma_n\| < \infty$. Then the Abel means $a_\lambda = \lambda \sum_0^{\infty} x_k/(\lambda + 1)^{k+1}$ $(\lambda > 0)$ exist and $\sup_{\lambda > 0} \|a_\lambda\| \leqslant M$.*

The proof is a routine exercise in partial summation but we include it for the sake of completeness. Set $a_k = (\lambda + 1)^{-(k+1)}$ in Abel's identity

$$\sum_{0}^{n} a_k x_k = \sum_{0}^{n-1} (a_k - a_{k+1})\, s_k + a_n s_n,$$

where $s_n = \sum_0^n x_j$, to obtain

$$\sum_0^n x_k/(\lambda+1)^{k+1} = \lambda \sum_0^{n-1} (k+1)\,\sigma_{k+1}/(\lambda+1)^{k+2} + (n+1)\,\sigma_{n+1}/(\lambda+1)^{n+1}.$$

Now let $n \to \infty$. Then

$$\sum_0^\infty x_k/(\lambda+1)^{k+1} = \sum_0^\infty (k+1)\,\sigma_{k+1}/(\lambda+1)^{k+2}$$

provided the series on the right converges. But this fact plus the rest of our theorem follow from absolute convergence, since **E** is complete:

$$\sum_0^\infty \frac{(k+1)\,\|\sigma_{k+1}\|}{(\lambda+1)^{k+2}} \leqslant M \sum_0^\infty \frac{k+1}{(\lambda+1)^{k+2}} = \lambda^{-2}M.$$

Hence the a_λ exist and $\|a_\lambda\| \leqslant M$ $(\lambda > 0)$.

THEOREM 3. *The Abel means $A(\lambda)(\lambda \to 0+)$ form a system of almost invariant integrals for T.*

Proof. Since $\lambda \sum_0^\infty (\lambda+1)^{-(n+1)} = 1$ $(\lambda > 0)$, condition (I) is clearly satisfied. (II) follows from Theorem 2. To establish (III) start from $A(\lambda) = \sum_0^\infty a_n T^n$, where $a_n = \lambda(\lambda+1)^{-(n+1)}$. The identity of Section 3 becomes $TA(\lambda) - A(\lambda) = \sum_0^\infty (a_n - a_{n+1})\,T^{n+1} - a_0 I = \lambda \sum_0^\infty (\lambda+1)^{-(n+2)}\,T^{n+1} - \lambda(\lambda+1)^{-1}I = \lambda A(\lambda) - \lambda^2(\lambda+1)^{-1}I - \lambda(\lambda+1)^{-1}I = \lambda[A(\lambda) - I]$. Hence $\|TA(\lambda) - A(\lambda)\| = \lambda\,\|A(\lambda) - I\| \leqslant \lambda(M+1) \to 0$ as $\lambda \to 0+$.

COROLLARY (Emilion's Theorem 2.1). *If E is reflexive* $\lim_{\lambda \to 0+} A(\lambda)x$ *exists for every x in E.*

To compare the $(C, 1)$ and Abel means, let

DEFINITION. $\mathscr{E} = [x \in \mathbf{E}: \lim_{n\to\infty} n^{-1}T^n x = 0]$. The identity $T_n - T_{n+1} = (n+1)^{-1}\,T_n - (n+1)^{-1}\,T^n$ implies $\mathscr{E} = [x \in \mathbf{E}\colon \lim_n (T_n - T_{n+1})\,x = 0]$. Because $\|T_n - T_{n+1}\| \leqslant 2M$, $\mathscr{E}$ is a closed subspace, clearly invariant under T, the T_n, and the $A(\lambda)$. Since the (T_n) act on $\mathscr{E}$ as almost invariant integrals, the Remark of Section 2 specializes to

THEOREM 4. *If $x \in \mathscr{E}$, $\lim_{\lambda\to 0+} A(\lambda)\,x = \lim_{n \to 0} T_n x$ in the sense that the existence of one of the limits implies the existence and equality of the other.*

Emilion [4] obtains a weaker conclusion (his Theorem 4.1) under the stronger hypotheses that **E** is reflexive and $\sup_n \|T^n x\| < \infty$. His method, due in fact to Hille [6], is to appeal to the following known

VECTOR TAUBERIAN THEOREM (Emilion's Theorem 3.1). *Let* (x_n) *be a bounded sequence in a Banach space and set* $a_\lambda = \lambda \sum_0^\infty x_n/(\lambda+1)^{n+1}$. *Then* $\lim_{\lambda\to 0+} a_\lambda = y$ *implies* $\lim_{n\to\infty} n^{-1}(x_0 + \cdots + x_{n-1}) = y$. (Abel convergence + boundedness implies $(C, 1)$ convergence.)

It is amusing that here the methods of so-called "hard" analysis yield a weaker result than those of so-called "soft" analysis. The time is long overdue to abandon these misleading catchwords.

5. EXTENSIONS

In a previous paper [3] we discussed the specialization of our abstract methods to the case of a continuous semi-group $T(t)$ $(t > 0)$. We shall leave it to the reader to derive strengthened forms of Emilion's results in this case. He assumes that $T(t)$ is strongly continuous, that $M = \sup_{t>0} t^{-1} \|S_t\| < \infty$, where $S_t x = \int_0^t T(s)\, x\, ds$, and sets $A(\lambda)\, x = \lambda \int_0^\infty e^{-\lambda s}\, T(s)\, x\, ds$ $(\lambda > 0)$. Most of his theorems seem to be contained in Hille and Phillips [6]. The relation of [6] to [2] is also of interest.

It would be a major task to survey all the mean ergodic literature since 1949 and see how much of it reduces to a specialization of our 1949 program. A great deal of this latter work has been supported by National Science Foundation (NSF) and other grants. Several years ago NSF declined to support our proposal to make such a survey.

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